

# A New Class of Rank Breaking Orbifolds

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## Abstract

We describe field-theory  $T^2/Z_n$  orbifolds that offer new ways of breaking  $SU(N)$  to lower rank subgroups. We introduce a novel way of embedding the point group into the gauge group, beyond the usual mapping of torus and root lattices. For this mechanism to work the torus Wilson lines must carry nontrivial 't Hooft flux. The rank lowering mechanism proceeds by inner automorphisms but is not related to continuous Wilson lines and does not give rise to any associated moduli. We give a complete classification of all possible  $SU(N)$  breaking patterns. We also show that the case of general gauge group can already be understood entirely in terms of the  $SU(N)$  case and the knowledge of standard orbifold constructions with vanishing 't Hooft flux.

# 1 Introduction

Orbifolds [1] are one of the most explored avenues in the study of string theory compactifications. Not only do they possess phenomenologically appealing features such as chirality, reduced supersymmetry, and a built-in gauge symmetry breaking mechanism, they are also extremely tractable and provide a welcome starting point to study more complicated vacua through string theory's many dualities. Notwithstanding, the classification of all orbifold vacua of the (heterotic) string seems to be an extremely difficult task, and the search of the standard model, or its supersymmetric extension, in this vast “landscape” of vacua has only been partially successful.

A more modest approach, justified in its own right, are orbifold grand unified theories (orbifold GUTs). It is quite conceivable that some of the extra dimensions are larger than others, and intermediate models with effectively fewer extra dimensions could be realized in nature. In view of this, a lot of effort has been made to construct five and six dimensional models that break the GUT group by orbifolding down to the SM [3–6]. Some intermediate 6d models appearing as particular compactification limits of the heterotic string have been described in Ref. [2].

A challenge in obtaining the standard model gauge group by orbifolding is the fact that the simplest consistent choices for the twists do not reduce the rank of the gauge group. In heterotic string theory, the anomaly-free gauge groups have rank 16 while the Standard Model only has rank 4. Rank reduction usually proceeds through one of the following mechanisms

- Continuous Wilson lines [19, 20]: A given orbifold vacuum can possess a nontrivial moduli space in the gauge sector, i.e. flat directions in the tree level potential for the extra dimensional components ( $A_{4,5,\dots}$ ) of the gauge bosons. The latter typically transform in non-adjoint representations of the gauge group left unbroken by the orbifolding. By obtaining vacuum expectation values they can break the gauge symmetries further, thereby reducing its rank. From a four dimensional (4d) point of view, this is nothing but the standard Higgs mechanism. This idea has been applied in the context of electroweak symmetry breaking and is often referred to as “gauge-Higgs unification” [7–16]. The flatness is lifted at loop level by a finite and calculable potential [8], giving rise to a discrete set of vacua. Unfortunately, in many circumstances, the vacuum calculated this way actually corresponds to a particular point

in moduli space where the rank of the gauge group is restored [11, 12]. Moreover, some Higgs mass terms localized at the fixed point are unprotected by the surviving gauge symmetry [13, 15] and can destroy the finiteness and predictivity of the model.

- Green-Schwarz mechanism: If the unbroken gauge group contains anomalous  $U(1)$  factors, the latter can be spontaneously broken by an orbifold version [17] of the Green-Schwarz mechanism [18]. This mechanism is realized, e.g., in the model of Ref. [10], where the rank-6 group  $U(3)^2$  was broken to the Standard model by the presence of two anomalous  $U(1)$  symmetries.
- Additional Higgs multiplets at the fixed points, as, e.g., in Ref. [4].
- Outer automorphisms. A particular choice of the gauge twists, corresponding to a symmetry of the Dynkin diagram of the associated Lie algebra, can break the rank. There are only finitely many possibilities.

In this paper we want to introduce a new way to break the rank of the gauge group by orbifolding. We will mainly restrict ourselves to  $T^2/Z_n$  orbifolds with gauge group  $SU(N)$  and will comment on generalizations to higher dimensional tori and other gauge groups in Sec. 4. An orbifold is specified by the gauge twists associated to translations and rotations of the underlying torus lattice. The spacetime translations commute, and so must the corresponding twists. However, in a pure gauge theory, the fields transform in the adjoint representation, and the twists need only commute up to an element of the center of the group. This yields nontrivial gauge bundles on the torus which still have a flat gauge connection (i.e. the corresponding field strength vanishes) [21]. The center of  $SU(N)$  is isomorphic to  $Z_N$ . Hence, there are  $N$  physically different disconnected vacua, or, more precisely, the moduli space consists of  $N$  disconnected components. The nontrivial statement we make in this paper is that one can orbifold these configurations. Since the distinction to the standard orbifold construction is quite essential, let us dwell a little more on this point. In the standard approach, lattice translations are realized by shift vectors, i.e. the corresponding holonomies exactly commute and can be realized as elements of the same Cartan torus. The rotations of the torus lattice are then realized by an element of the Weyl group (rotations of the root lattice). Here, instead, the lattice translations are already realized as rotations of the root lattice, in a way that makes it

impossible to choose a Cartan torus such that both of them simultaneously become shifts. Consequently, the orbifold twists associated to the rotations of the torus lattice cannot be related to any symmetry of the root lattice used to define the torus holonomies.

The paper is organized as follows. In Sec. 2 we review the nontrivial flat  $SU(N)$  gauge bundles on the two-torus, give an explicit form for the holonomies, and describe their symmetry breaking patterns. We also explain how other gauge groups can be treated once the  $SU(N)$  case is known. These gauge bundles are orbifolded in Sec. 3. In Sec. 3.1 we treat first the case  $m = 0$ . This does not involve any new concepts, but we include it here for completeness and comparison. Also, in App. B we compute the moduli space for this case. In Sec. 3.2 we calculate the orbifold twists for the generic case, making use of the results obtained in Sec. 2 and Sec. 3.1. Finally, in Sec. 4 we summarize our results and discuss some applications.

## 2 Breaking $SU(N)$ on $T^2$ : Torons.

In this section we would like to recall 't Hooft's toron configurations [21]. These are simply flat  $SU(N)$  gauge bundles on the torus, which can be characterized by their holonomies. Upon shifts in the torus lattice

$$z \rightarrow z + \lambda \tag{2.1}$$

gauge fields are identified up to gauge transformations<sup>1</sup>

$$A_M(z + \lambda) = T_\lambda A_M(z) T_\lambda^{-1}. \tag{2.2}$$

It is clearly sufficient to restrict to the two lattice-defining base vectors  $\lambda_{1,2}$ . As lattice translations commute, the commutator of the two transition functions has to act as the identity.

$$T_1 T_2 T_1^{-1} T_2^{-1} = e^{2\pi i \frac{m}{N}}. \tag{2.3}$$

On the right hand side we have allowed for a general element of the center of the group, which, for  $SU(N)$ , equals  $Z_N$ . Such a gauge transformation indeed acts trivially on the adjoint representation the gauge fields transform

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<sup>1</sup>We make use of the fact that we can choose a gauge where the transition functions are  $z$ -independent, see, e.g., Ref. [22].

in. The integer quantity  $m$  is called the 't Hooft nonabelian flux. We stress that it is in principle possible to simultaneously diagonalize the matrices  $T_1$  and  $T_2$  in the adjoint.<sup>2</sup> For nonzero  $m$ , it is not possible to represent both  $T_i$  as elements of the same Cartan torus. It is, however, possible to choose a Cartan torus left fixed (though not pointwise fixed) by both  $T_i$ . As a consequence, one can realize the  $T_i$  as Weyl group elements w.r.t. the same Cartan subalgebra.

The flux  $m$  (more precisely the phase appearing on the r.h.s. in Eq. (2.3)) labels the equivalence classes of the transition functions and determines the vacua of the theory. We would like to find the unbroken subgroup for each vacuum, i.e. we are looking for the generators that are left invariant by the action of the  $T_i$ :

$$T_i \mathcal{T} T_i^\dagger = \mathcal{T}. \quad (2.4)$$

For fixed  $m$ , there is still a continuous degree of freedom in choosing the  $T_i$ , even within the gauge where the transition functions are constants: If, for a particular solution to Eq. (2.3), the unbroken subgroup  $\mathcal{H}$  is nontrivial, one can always turn on Wilson lines in the Cartan torus of  $\mathcal{H}$  and still obtain a solution with the same value for  $m$ . Such an additional Wilson line will lead to a different subgroup  $\mathcal{H}'$ , however, the rank of  $\mathcal{H}$  and  $\mathcal{H}'$  must remain the same. This freedom is related to the fact that each vacuum will in general possess a nonzero moduli space, i.e. flat directions in the potential for the extra dimensional components of  $A$ .

To describe the solutions, one decomposes  $N$  and  $m$  according to their greatest common divisor  $K = \text{g.c.d}(N, m)$ . Explicit solutions to Eq. (2.3) are then given by [22, 23]

$$T_1 = Q_{N/K} \otimes \mathbb{1}_K, \quad (Q_L)_{jk} = q_L^{-(L-1)/2} \delta_{j,k-1}, \quad (2.5)$$

$$T_2 = (R_{N/K})^{m/K} \otimes \mathbb{1}_K, \quad (R_L)_{jk} = q_L^{-(L-1)/2+j-1} \delta_{j,k}, \quad (2.6)$$

where  $q_L = \exp(2\pi i/L)$ . The index on  $Q$ ,  $R$  and  $\mathbb{1}$  indicates the dimensionality of the matrices and the Kronecker  $\delta$  is assumed to be periodic. The matrices  $Q$  and  $R$  satisfy

$$Q R = q R Q, \quad Q^L = R^L = (-)^{L-1} \mathbb{1}. \quad (2.7)$$

Hence, Eq. (2.5) and Eq. (2.6) are a particular solution to Eq. (2.3). It can then be shown that the twists  $Q_{N/K}$  and  $(R_{N/K})^{m/K}$  break  $SU(N/K)$

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<sup>2</sup>For an explicit diagonal basis see Ref. [22].

completely [22]. Writing the generators of  $SU(N)$  as

$$\mathcal{T}_N \in \{\mathcal{T}_{N/K} \otimes \mathcal{T}_K, \mathbb{1}_{N/K} \otimes \mathcal{T}_K, \mathcal{T}_{N/K} \otimes \mathbb{1}_K\}. \quad (2.8)$$

We immediately read off that the unbroken subgroup is generated by  $\mathbb{1}_{N/K} \otimes \mathcal{T}_K$  and, thus, is  $SU(K)$ . The most general solution to Eq. (2.3) can then be obtained by replacing the unit matrices in Eq. (2.5) and (2.6) with commuting Wilson lines of  $SU(K)$ , which one can take to be elements of the same Cartan torus:

$$\begin{aligned} T_1 &= Q_{N/K} \otimes \exp(2\pi i W_1), \\ T_2 &= (R_{N/K})^{m/K} \otimes \exp(2\pi i W_2). \end{aligned} \quad (2.9)$$

The shift vectors  $W_1$  and  $W_2$  are elements of the Cartan subalgebra of  $SU(K)$ . Nontrivial  $SU(K)$  Wilson lines further break  $SU(K)$ , but do not reduce its rank. In summary, a toron configuration with  $SU(N)$  flux  $m$  can be decomposed into a toron configuration with  $SU(N/K)$  flux  $m/K$  and an  $SU(K)$  configuration with vanishing flux.

However, we would like to stress here that different  $SU(K)$  Wilson lines, strictly speaking, do not correspond to different physical theories. The reason is that the above mentioned flat directions are lifted at the quantum level and two such theories will dynamically evolve to the same vacuum. One can always perform a field redefinition, corresponding to a nonperiodic gauge transformation that removes the continuous Wilson line but generates a vacuum expectation value (VEV)

$$\begin{aligned} A_4 &= W_1, \\ A_5 &= W_2. \end{aligned} \quad (2.10)$$

One sees that such a field redefinition induces a shift along a flat direction. In other words, a theory with nonzero Wilson line and a given point in the moduli space is equivalent to a vanishing Wilson line and a shifted point in moduli space. The degeneracy of the flat directions is lifted at the quantum level. The effective potential clearly only depends on the sum of the Wilson line induced background, Eq. (2.10), and the explicit background, and the true vacuum of two theories with different continuous Wilson lines coincide. It is important to realize that there is no analogous field redefinition that could change the value of  $m$ :<sup>3</sup> Two vacua with different  $m$  are truly disconnected.

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<sup>3</sup>By enforcing such a field redefinition to, say, remove the Wilson line  $T_1$ , the other transition function would no longer remain constant.

The natural question to ask is whether all this can be generalized to gauge groups other than  $SU(N)$ . This question has been extensively discussed in Ref. [24], see also Refs. [25–27]. Here we only give some heuristic arguments and some examples. A necessary and sufficient condition for the existence of nontrivial 't Hooft flux is that the group possesses nontrivial center.<sup>4</sup> This is true for the  $SO(N)$  and  $Sp(2N)$  groups, as well as for the exceptional groups  $E_6$  and  $E_7$ . The center can always be embedded in suitable  $SU(N)$  subgroups [24], and, hence, the above construction can be carried out straightforwardly. Trivial examples are the groups  $SO(3)$ ,  $SO(4)$ ,  $SO(6)$ , and  $Sp(2)$  that are actually isomorphic to some special unitary groups. For a nontrivial example take  $SO(8)$  whose center is  $C = Z_2 \times Z_2$ . Consider now the maximal subgroup  $SU(2)^4 \subset SO(8)$ . By inspection of the branching rules for the  $SO(8)$  irreducible representations  $\mathbf{8}_v$  and  $\mathbf{8}_s$ , one can see that a suitable parametrization of the two  $Z_2$ 's of the center is

$$c_1 = (-\mathbb{1}, -\mathbb{1}, \mathbb{1}, \mathbb{1}), \quad c_2 = (\mathbb{1}, -\mathbb{1}, -\mathbb{1}, \mathbb{1}), \quad (2.11)$$

where  $\pm\mathbb{1}$  represent the center of the corresponding  $SU(2)$  factor. The branching of the adjoint is

$$\mathbf{28} \rightarrow (\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}). \quad (2.12)$$

It can be directly verified that  $C$  acts trivially on the  $\mathbf{28}$ , as it must. For a given  $c \in C$ , particular solutions for  $T_1(c)$  and  $T_2(c)$  can now be constructed by making use of the results for  $SU(2)$ . While any pair  $T_i(c)$  clearly projects out two of the four triplets, the action on the fourfold doublet requires a more careful analysis. Take, for instance  $c = c_1$ , then the standard solution acts on the  $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})$  as

$$T_1 = \sigma_1 \otimes \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1}, \quad T_2 = \sigma_3 \otimes \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1}. \quad (2.13)$$

The two twists can be diagonalized simultaneously. There are four eigenstates, each transforming as  $(\mathbf{2}, \mathbf{2})$  of the surviving  $SU(2)^2$ . One of these eigenstates has unit eigenvalue on both  $T_i$ , and hence the branching rule of the adjoint under the breaking  $SO(8) \rightarrow SU(2)^2$  reads

$$\mathbf{28} \rightarrow (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{2}), \quad (2.14)$$

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<sup>4</sup>More precisely, the center of the universal cover, which is isomorphic to the fundamental group of the adjoint representation.

corresponding to the breaking

$$SO(8) \rightarrow SO(5). \quad (2.15)$$

It is remarkable that we can obtain a non-regular subgroup of  $SO(8)$  by the combination of two *inner* automorphisms of  $SO(8)$ . Each twist  $T_i$  breaks  $\mathcal{G} = SO(8)$  to a regular subgroup  $\mathcal{H}_i$  (in this case  $SU(4) \times U(1)$ ). However,  $T_2$  is not contained in  $\mathcal{H}_1$ , and, although being an inner automorphism on  $\mathcal{G}$ , it acts as an outer automorphism on  $\mathcal{H}_1$ . The result is the special subgroup  $SO(5)$  of  $SU(4) \times U(1)$ . All other gauge groups can, in principle, be calculated along these lines. For a list of gauge groups that can be obtained this way we refer the reader to Tab. 6 in Ref. [27].

### 3 Breaking $SU(N)$ on the orbifold

The torus lattice has a discrete rotational symmetry that can be modded out to obtain the  $T^2/Z_n$  orbifold. The only discrete rotations possible are of order  $n = 2, 3, 4, 6$ . The topology of the resulting spaces are “pillows”, see Fig. 1. The two sides of the pillow represent the bulk and the corners the fixed points. We depict the four possibilities in Fig. 1. Notice that  $T^2/Z_4$  contains two  $Z_4$  and one  $Z_2$  singularity and  $T^2/Z_6$  contains one  $Z_2$ ,  $Z_3$  and  $Z_6$  singularity each.

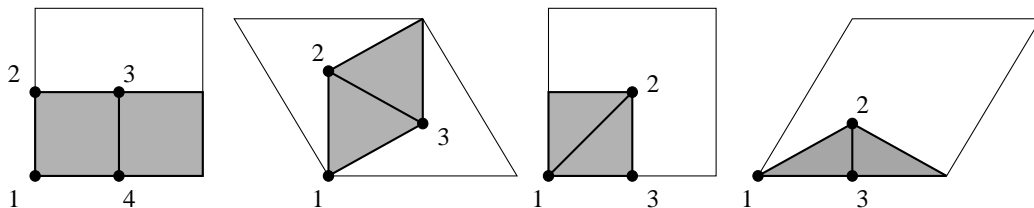


Figure 1: The four different  $Z_n$  orbifold geometries in 6d, corresponding to  $n = 2, 3, 4, 6$  (from left to right). We show the embedding of the orbifold fundamental domain (shaded) in the torus (thin line) as well as the fixed points (dots). The shaded regions have to be folded over the center line and the edges (thick lines) have to be identified. The resulting geometries are “pillows” with three or four corners. Note that the edges correspond to non-singular bulk points.



In analogy to Eq. (2.2), one now introduces orbifold twists

$$A_M(pz) = P A_M(z) P^{-1}. \quad (3.1)$$

where  $p$  is the  $n$ th root of unity

$$p = \exp(2\pi i/n). \quad (3.2)$$

The additional identification leads to new constraints. Besides the obvious,  $p^n = 1$ , one also has to take into account that a  $Z_n$  rotation followed by a translation along some lattice vector, followed again by the inverse rotation, equals a lattice translation along the rotated vector:

$$p^{-1}(pz + \lambda) = z + p^{-1}\lambda. \quad (3.3)$$

The full set of constraints is thus <sup>5</sup>

$$T_\lambda T_{\lambda'} \sim T_{\lambda'} T_\lambda \sim T_{\lambda+\lambda'}, \quad (3.4)$$

$$P^{-1} T_\lambda P \sim T_{p\lambda}, \quad (3.5)$$

$$P^n \sim \mathbb{1}. \quad (3.6)$$

Here we have introduced the equivalence relation  $\sim$  defined as “equal modulo an element of the center of  $SU(N)$ ”.<sup>6</sup> The most general solution to the first of these constraints has been presented in the previous section. The main purpose of this paper is to show that there are nontrivial solutions to the other two constraints, given torus Wilson lines with generic  $m$  and for any  $n = 2, 3, 4, 6$ .

There is an alternative description to Eq. (3.4) to Eq. (3.6), called the downstairs picture, that only makes reference to the fundamental domain of the orbifold (i.e. the physical space). For any given fixed point  $z_f$  of the rotation  $p^k$ , one can define a rotation around  $z_f$ :

$$p_{z_f}(z) = p^k z + \lambda, \quad \lambda = (1 - p^k)z_f \quad (3.7)$$

where  $\lambda$  is a lattice vector. Choosing any fundamental orbifold domain, the product over the rotations around all four ( $Z_2$ ) or three ( $Z_{3,4,6}$ ) fixed points

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<sup>5</sup>For the second relation, notice that the order of the gauge group elements is reversed w.r.t. the space group.

<sup>6</sup>Mathematically speaking we are looking for special unitary projective representations of the space group.

equals a pure lattice translation, and some special combinations even yield the trivial one: choosing the fundamental domains and fixed points labels as in Fig. 1, one finds

$$p_{z_4}p_{z_3}p_{z_2}p_{z_1} = 1, \quad n = 2, \quad (3.8)$$

$$p_{z_3}p_{z_2}p_{z_1} = 1, \quad n = 3, 4, 6. \quad (3.9)$$

Obviously, any cyclic permutation of these relations hold. For  $n = 2$ , the anticyclic order also yields one (but not an arbitrary permutation),<sup>7</sup> while for  $n = 3, 4, 6$  the anticyclic order already yields a nontrivial shift. Again, these relations must be represented by the corresponding twists:

$$\begin{aligned} P_{z_1}P_{z_2}P_{z_3}P_{z_4} &\sim \mathbb{1}, & n = 2, \\ P_{z_1}P_{z_2}P_{z_3} &\sim \mathbb{1}, & n = 3, 4, 6, \\ (P_{z_i})^{\nu_i} &\sim \mathbb{1}, \end{aligned} \quad (3.10)$$

with  $\nu_i$  being the order of the fixed point  $z_i$ . By re-expressing the lattice shifts through the rotations, it can be shown that, conversely, the relations Eq. (3.10) imply Eqns. (3.4) to (3.6). In other words, the downstairs picture (in which we specify the local orbifold twists) is completely equivalent to the upstairs picture (in which we specify the torus Wilson lines and the basic  $Z_n$  orbifold twist). Moreover, the downstairs relations can be further reduced by actually solving Eq. (3.10) for one of the twists in terms of the others. In the case of  $Z_6$ , for instance, the relations then reduce to

$$P_{z_2}^3 \sim \mathbb{1}, \quad P_{z_3}^2 \sim \mathbb{1}, \quad (P_{z_2}P_{z_3})^6 \sim \mathbb{1}. \quad (3.11)$$

While the first two relations are always easy to satisfy, the last relation becomes highly nontrivial if the two twists do not commute. In fact, the product  $P_{z_3}P_{z_2}$  does not even have to have finite order. It is possible to generalize the orbifold construction to allow for gauge twists whose order does not match that of the spacetime twist [5]. Such models then allow for many more rank breaking possibilities. While the downstairs picture is very useful, in particular for the case of commuting Wilson lines, in this paper we will mainly stick to the upstairs description. For one, it makes an important aspect of the new rank breaking mechanism manifest: it can be viewed as

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<sup>7</sup>This can easily be seen by taking the inverse of Eq. (3.8) and using the fact that  $p_{z_f}^2 = 1$ .

an orbifold of topologically nontrivial torus Wilson lines. Secondly, the nice factorization of the torus Wilson lines, obvious from Eq. (2.9), carries over to the orbifold twists and presents a convenient way to classify all possible orbifolds.

The rest of this section is organized as follows. In Sec. 3.1 we will calculate the  $SU(N)$  breaking on the orbifold in the case of vanishing 't Hooft flux. In particular, we will focus on breakings by continuous Wilson lines, corresponding to the part of the moduli space of the torus that survives the orbifold projection. In Sec. 3.2 we will then show how to construct orbifold twists that fulfill Eq. (3.5) and Eq. (3.6) for generic  $m$  and  $N$ .

### 3.1 The case $m = 0$

In the case  $m = 0$ , there exists a well defined scheme [19] to construct solutions to Eq. (3.5) and (3.6), by identifying  $P$  with a suitable element of the Weyl group, the symmetry group of the root lattice of the Lie algebra. Such an element induces an algebra automorphism that maps the Cartan subalgebra onto itself. For a given orbifold twist, the Cartan subalgebra naturally decomposes into two subspaces: The eigenspaces to unit and non-unit eigenvalues under the linear map  $P$ . The latter give rise to Wilson lines that commute with  $P$ , and Eq. (3.5) implies that they are discrete. For  $n = 2$  one finds

$$T_1^2 = T_2^2 = \mathbb{1}, \quad (3.12)$$

while for  $n = 3, 4$  one has  $T_1 = T_2 \equiv T$ , with

$$\begin{aligned} T^3 &= \mathbb{1}, & n &= 3, \\ T^2 &= \mathbb{1}, & n &= 4. \end{aligned} \quad (3.13)$$

For  $n = 6$  there are no discrete Wilson lines.<sup>8</sup> Wilson lines not invariant under  $P$  can still exist and can be constructed as follows. Consider the shift vector as a map from the torus lattice to the root lattice, then we can rewrite Eq. (3.5) as a composition of maps

$$P_{adj}^{-1} \circ V = V \circ p. \quad (3.14)$$

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<sup>8</sup>In the downstairs picture this can also be easily understood: any commuting triple fulfilling Eq. (3.10) in the case  $n = 6$  automatically also satisfies  $P_{z_2} = P_{z_1}^2$ ,  $P_{z_3} = P_{z_1}^3$ . Hence, the  $Z_6$  twist  $P_{z_1}$  already determines the other two twists.

If the torus lattice can be embedded into the root lattice of the algebra, one can choose  $V$  to be any scalar multiple of that embedding and identify the rotation  $P^{-1}$  with  $p$ . The Wilson lines defined this way are thus continuous and will break the rank [19, 20]. We will not make use of this description in this paper. Rather, we will consider an equivalent description in terms of the zero modes of  $A_{4,5}$ . Just in the case of the torus, the continuous Wilson lines can be transformed into background VEVs for these extra dimensional components of the gauge bosons and, hence, parametrize the moduli space of the compactification. The advantage of this approach is that we can represent  $P$  as a shift (element of the Cartan torus) rather than a rotation (element of the Weyl group).

Let us consider the case that the gauge twist is the same at each fixed point (no discrete Wilson lines). The orbifold shift vector can be taken, without loss of generality, to be of the form [31]

$$V = \frac{1}{n}(k_1, k_2 \dots k_r), \quad k_i > 0, \quad \sum_i k_i \leq n-1. \quad (3.15)$$

As shown in App. B, a flat direction exists if and only if there are exactly  $n-1$  entries with  $k_i = 1$  with the remaining  $k_i = 0$ :

$$V = \frac{1}{n}(0^{r_1}, 1, 0^{r_2}, 1, \dots, 0^{r_{n-1}}, 1, 0^{r_n}). \quad (3.16)$$

Here,  $0^r$  stands for an  $r$  dimensional zero vector (some of the  $r_i$  may be zero). Notice that this means, in particular, that the inequality in Eq. (3.15) is saturated. For  $n=2$ , Eq. (3.15) already implies a shift vector that is either trivial or of the form Eq. (3.16) and, hence, there are always flat directions for nontrivial  $V$ . For generic  $n$ , the breaking pattern induced by this shift vector is

$$SU(N) \rightarrow \mathcal{H}_0 \equiv \prod_{i=1}^n SU(N_i) \times U(1)^{n-1}, \quad \sum_{i=1}^n N_i = N, \quad (3.17)$$

with  $N_i = r_i + 1$ . There are  $N_{min} = \min\{N_i\}$  flat directions, which are calculated in App. B. There it is shown that, for vanishing discrete Wilson lines, a generic point in moduli space breaks  $SU(N)$  according to

$$SU(N) \rightarrow \mathcal{H} \equiv \prod_{i=1}^n U(N_i - N_{min}) \times U(1)^{N_{min}-1}. \quad (3.18)$$

with

$$N \geq n, \quad N_k \geq 1, \quad N_{min} = \min\{N_k\}, \quad \sum_{k=1}^n N_k = N. \quad (3.19)$$

The rank of  $SU(N)$  is reduced by  $N_{min}(n-1)$ .

To complete the classification, one could turn on discrete Wilson lines. The full moduli space of the  $T_i = \mathbb{1}$  case survives this additional projection if and only if the  $T_i$  reside in the Cartan torus of  $\mathcal{H}$ . In this case, the unbroken subgroup can be any full-rank subgroup of  $\mathcal{H}$ . It is possible that only a subspace of the moduli space survives. However, a complete treatment of these cases lies outside the scope of the present paper and we will omit it here for brevity. For  $Z_6$  there are no discrete Wilson lines, and our analysis already covers all possible breaking patterns. The smallest group whose rank can be spontaneously broken in a  $Z_6$  orbifold (with vanishing  $m$ ) is thus  $SU(6)$ , with a single modulus breaking all of  $SU(6)$ .

### 3.2 Generic $m$

Our classification of solutions to Eqns. (3.5) and (3.6) for generic  $m$  proceeds in two steps. First, we construct the solution  $P_{N,m}$  for  $m, N$  coprime, which always breaks  $SU(N)$  completely, as we have seen in Sec. 2. For arbitrary  $(N, m)$ , we write the most general solution as

$$T_1 = Q_{N/K} \otimes \exp(2\pi i W_1), \quad (3.20)$$

$$T_2 = (R_{N/K})^{m/K} \otimes \exp(2\pi i W_2), \quad (3.21)$$

$$P = P_{N/K, m/K} \otimes \exp(2\pi i V). \quad (3.22)$$

where  $W_i$  are discrete Wilson lines subject to Eqns. (3.12) and (3.13). The shift vectors  $V$  and  $W_i$  are elements of the Cartan subalgebra of  $SU(K)$ . The moduli space of this geometry is then given by the moduli space of an  $SU(K)$  theory with vanishing flux. For trivial discrete Wilson lines, this moduli space has been given in Sec. 3.1 and App. B.

It remains to be shown that, given the Wilson lines <sup>9</sup>

$$T_1 = Q, \quad T_2 = R^m, \quad (3.23)$$

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<sup>9</sup>We will drop the indices  $N$  and  $m$  for the rest of the section.

for  $m$  and  $N$  coprime, we can actually construct an orbifold twist  $P$  that fulfills Eq. (3.5) and (3.6). For  $n = 2$ , it is very easy to write down such a  $P$ . The matrix

$$P_{k\ell} = \delta_{k,-\ell} \quad (3.24)$$

can easily be confirmed to fullfill the requirements. For  $n = 3, 4, 6$ , we can choose our lattice to be generated by  $\lambda_1 = 1$  and  $\lambda_2 = p$ . Relation Eq. (3.5) then implies that for any  $n = 3, 4, 6$ , we must have

$$P^{-1}QP \sim R^m \quad (3.25)$$

as well as

$$PQP^{-1} \sim \begin{cases} R^{-m}Q^{-1} & n = 3 \\ R^{-m} & n = 4 \\ R^{-m}Q & n = 6 \end{cases} \quad (3.26)$$

The matrices  $Q$  and  $R$  have the same eigenvalues, given by the  $N$  different  $N$ th roots of unity. As  $m$  and  $N$  are coprime, the same holds true for  $R^m$ . As a consequence, one can always find an  $SU(N)$  matrix  $U$  that satisfies

$$UQU^\dagger \sim R^m. \quad (3.27)$$

We choose  $U$  as <sup>10</sup>

$$U_{k\ell} = N^{-\frac{1}{2}} q^{-(k-1)\ell m}, \quad q = e^{\frac{2\pi i}{N}} \quad (3.28)$$

The proof that  $U$  indeed satisfies Eq. (3.27) is presented in App. A. Moreover,  $U$  also satisfies

$$U^\dagger QU \sim R^{-m}. \quad (3.29)$$

Notice that  $U$  can be multiplied by any diagonal  $SU(N)$  matrix from the left without affecting Eq. (3.27), as  $R$  is diagonal. However, Eq. (3.29) will be modified. It can be shown that there is a diagonal  $SU(N)$  matrix  $X$  satisfying <sup>11</sup>

$$XQX^\dagger \sim QR^m \quad \Leftrightarrow \quad X^\dagger QX \sim QR^{-m}. \quad (3.30)$$

Multiplying  $U$  with  $X$  we find

$$(XU)^\dagger Q(XU) \sim U^\dagger QR^{-m}U \sim R^{-m}U^\dagger R^{-m}U \sim R^{-m}Q^{-1} \quad (3.31)$$

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<sup>10</sup>This matrix is known as a Vandermonde matrix. The matrix in Eq. (3.28) should be divided by its determinant to obtain an  $SU(N)$  matrix, which we omit here for clarity.

<sup>11</sup>We give the precise form of  $X$  in App. A.

where in the first step we used Eq. (3.30), in the second step Eq. (3.29) and in the last one Eq. (3.27). In a completely analogous fashion one can show that

$$(X^\dagger U)^\dagger Q (X^\dagger U) \sim R^{-m} Q. \quad (3.32)$$

One concludes that by choosing

$$P^{-1} = \begin{cases} XU & n = 3 \\ U & n = 4 \\ X^\dagger U & n = 6 \end{cases} \quad (3.33)$$

we satisfy both Eq. (3.27) and Eq. (3.29). It remains to be shown that

$$(XU)^3 \sim \mathbb{1}, \quad U^4 \sim \mathbb{1}, \quad (X^\dagger U)^6 \sim \mathbb{1}. \quad (3.34)$$

We again postpone the proof of this to App. A.

Let us illustrate these general considerations with the simplest possible example:  $SU(2)$ . The only possible nontrivial choice is  $m = 1$ . In the adjoint the two Wilson lines read:

$$T_1 = \text{diag}(+1, -1, -1), \quad T_2 = \text{diag}(-1, -1, +1). \quad (3.35)$$

For the  $Z_2$  case, Eq. (3.24) actually gives the identity for  $P$ . It follows that in this case the local twists are simultaneously diagonal in the adjoint:

$$\begin{aligned} P_{z_1} &= \text{diag}(+1, +1, +1), \\ P_{z_2} &= \text{diag}(-1, -1, +1), \\ P_{z_3} &= \text{diag}(-1, +1, -1), \\ P_{z_4} &= \text{diag}(+1, -1, -1). \end{aligned} \quad (3.36)$$

This only happens in the case  $N = n = 2$ . At one fixed point  $SU(2)$  is left unbroken, while at every other fixed point a different  $U(1)$  survives. Note that this breaking pattern is qualitatively different from the usual breaking of  $SU(2)$  by continuous Wilson lines, as described in Sec. 3.1: There the local gauge group is  $U(1)$  at all four fixed points. For  $Z_3$  we find for the twist

$$P = P_{z_1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (3.37)$$

The local twists are now truly non-commutative as can be seen by computing the twists associated to the other two fixed points:

$$PT_1T_2 = P_{z_2} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad PT_1 = P_{z_3} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \quad (3.38)$$

Geometrically, these twists are  $SO(3)$  rotations by  $120^\circ$  around the axes  $(1, 1, 1)$ ,  $(1, -1, -1)$  and  $(-1, -1, 1)$  respectively. Each axis of rotation defines a  $U(1)$  subgroup that remains unbroken at the corresponding fixed point. It is easy to verify that the product  $P_{z_1}P_{z_2}P_{z_3}$  indeed gives the identity.

In summary, we have seen that the discrete torus Wilson lines that break  $SU(N)$  down to  $SU(K)$ , with  $K$  any divisor of  $N$ , are orbifold compatible, i.e. there exists an orbifold twist that fulfills Eq. (3.5) and Eq. (3.6), for any  $n = 2, 3, 4, 6$ . In comparison to the mechanism of rank reduction described in Sec. 3.1, there are no moduli associated to this breaking. Before concluding this section we would like to comment on the inclusion of matter to this scenario. Up to now, we have only considered pure gauge theory or, more precisely, only fields in the adjoint of the group. Matter usually transforms in representations that are sensitive to the center of the group (such as the fundamental) and, hence, potentially destroy some or all of the torus configurations. On the orbifold it is not uncommon that non-adjoint matter only appears on the fixed points (as, e.g., in constructions that have extended  $\mathcal{N} = (1, 1)$  supersymmetry in 6d). Another possibility is to include other global or local symmetries in the twists to compensate for the nontrivial action of the center.

## 4 Discussion and Conclusions

In this paper we have analyzed orbifolds that break the gauge group  $SU(N)$  to lower rank subgroups. The rank breaking proceeds through nontrivial toron configurations, meaning the gauge fields have twisted boundary conditions on the covering torus of the orbifold. These twisted boundary conditions are of topological nature, characterized by the 't Hooft flux, and, as a consequence, they cannot be transformed into a constant background VEV for any extra dimensional components of the gauge fields. The main result of this paper is that one can actually orbifold these configurations and that



a classification of all possible breakings emerges from this approach. Torus Wilson lines can break  $SU(N)$  down to  $SU(K)$ , where  $K$  is a divisor of  $N$ . The orbifold is compatible with such a breaking, and the remaining freedom in choosing the orbifold twists is that of an orbifold with  $SU(K)$  gauge group and trivial (commuting) torus Wilson lines.

As mentioned at the end of Sec. 2, this result can be generalized almost straightforwardly to the case of other gauge groups with nontrivial center: the center can be embedded in suitable  $SU(N)$  subgroups and the construction of torus and orbifold twists proceeds as before. They leave an unbroken subgroup that can be orbifolded in the standard way (i.e., with continuous and discrete Wilson lines in the topologically trivial sector). As a matter of fact, the centers of groups other than  $SU(N)$  are given by abelian groups of order  $\leq 4$ . Hence, the corresponding twists are particularly simple: they just correspond to the  $SU(N)$  twists described in this paper with  $N \leq 4$ . A more careful treatment of general gauge groups is postponed to a future publication.

Another possible generalization concerns higher dimensional orbifolds (based on tori  $T^d$  with  $d > 2$ ). For  $d > 2$ , the fundamental group of the adjoint (or, equivalently, the center of the universal cover) is no longer sufficient to characterize the flat connections on the torus. In fact, for  $SO(N)$  with  $N \geq 7$ , as well as all exceptional groups, there do exist commuting triples that cannot be simultaneously conjugated to the same Cartan torus [24, 27–29]. The surviving unbroken subgroup is therefore rank-reduced. For instance, the exceptional group  $E_8$ , which does not have nontrivial pairs, nevertheless possesses nontrivial triples. It would therefore be interesting to construct orbifolds based on these nontrivial torus vacua.<sup>12</sup>

One can, however, immediately apply our results to 10d orbifolds by considering particular compactification limits. Take a heterotic orbifold with visible gauge group  $E_8$ . One can think of compactifying two of the three two-tori, leaving over an effective 6d theory. It is certainly possible, by making use of standard rank preserving orbifold breakings, to break  $E_8$  to the subgroup  $SO(10) \times SU(4)$  in 6d. In a second step, we break the  $SU(4)$  factor completely with our mechanism, while, at the same time, use the 6d orbifold to construct a realistic  $SO(10)$  orbifold GUT model. It is also possible to break to a 6d theory with gauge group  $E_6 \times SU(3)$ . Standard rank-breaking

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<sup>12</sup>Note that the asymmetric orbifolds of Refs. [30] are not related to our construction. The twists employed here can only correspond to symmetric orbifolds in string theory.

mechanisms might be used to get the Standard Model from  $E_6$  [20], while the additional “flavor”  $SU(3)$  can be broken by the methods described in this paper. A more direct application would be an orbifold reduction of the  $SO(32)$  heterotic string to eight dimension. The  $T^2$  compactification has been described in Ref. [26, 28], leading to  $Sp(16)$  gauge symmetry in 8d.

Last but not least we would like to comment on an application to supersymmetry breaking in six dimensions. Minimal  $\mathcal{N} = (1, 0)$  supersymmetry has an  $R$ -symmetry group  $SU(2)_R$ . One can break  $SU(2)_R$  and, hence, supersymmetry completely by continuous Wilson lines in the case of  $Z_2$  orbifolds but not for  $Z_{3,4,6}$ . We have shown that it is nevertheless possible to find discrete Wilson lines that break all of  $SU(2)_R$  for arbitrary  $Z_n$ , and such a Scherk-Schwarz mechanism is possible. Within this context it is interesting to notice that no continuous parameter exists that controls supersymmetry breaking, yet the breaking is still soft, as locally at least  $N = 1$  supersymmetry is preserved at all fixed points. Similar constructions can of course be applied to break all or part of extended supersymmetry.

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## A Some technicalities

In this appendix we will prove Eqns. (3.27), (3.29), (3.30), and (3.34). Throughout this section  $m$  and  $N$  are coprime integers and  $q$  is defined as

$$q = e^{\frac{2\pi i}{N}}. \quad (\text{A.1})$$

Using the property

$$\sum_{k=1}^N q^{\ell k} = N \delta_{\ell,0}, \quad (\text{A.2})$$

Eq. (3.27) and Eq. (3.29) can be readily verified:

$$\begin{aligned}
(UQU^\dagger)_{k\ell} &= \frac{q^{-\frac{N-1}{2}}}{N} \sum_{i,j} q^{-(k-1)i + j(\ell-1)m} \delta_{i,j-1} \\
&= \frac{q^{(k-1)m - \frac{N-1}{2}}}{N} \sum_j ((q^m)^{\ell-k})^j = q^{(k-1)m - \frac{N-1}{2}} \delta_{k\ell} \\
&= q^{\frac{(N-1)(m-1)}{2}} (R^m)_{k\ell} \sim (R^m)_{k\ell}
\end{aligned} \tag{A.3}$$

In the last step of the second line we have made use of the fact that Eq. (A.2) holds if  $q$  is replaced with  $q^m$  for  $m$  and  $N$  coprime. In the last step we have used that  $(N-1)(m-1)$  is always even. The proof of Eq. (3.29) is completely analogous and we will skip it here.

The identity  $U^4 \sim U^{\dagger 4} \sim 1$  is also quite easy. For  $m = 1$

$$\begin{aligned}
(U^\dagger)_{k\ell}^4 &= \frac{1}{N^2} \sum_{i,j,r} q^{k(i-1) + i(j-1) + j(r-1) + r(\ell-1)} \\
&= \frac{1}{N^2} \sum_{i,r} q^{ki - k - i + r\ell - r} \sum_j (q^{i+r-1})^j \\
&= \frac{1}{N} \sum_{i,r} q^{ki - k - i + r\ell - r} \delta_{i,1-r} \\
&= \frac{\bar{q}}{N} \sum_r (q^{\ell-k})^r = \bar{q} \delta_{k\ell}.
\end{aligned} \tag{A.4}$$

For  $m \neq 1$  just replace  $q \rightarrow q^m$ . Let us now define

$$X_{k\ell} = q^{-\frac{k(k-N)}{2}} \delta_{k\ell}. \tag{A.5}$$

The matrix  $X$  does not have unit determinant,  $\det X = e^{-\pi i \frac{N^2-1}{3}}$ . As in the case of  $U$ , this can easily be cured by a rescaling. Now calculate:

$$\begin{aligned}
(XQX^\dagger)_{k\ell} &= q^{-\frac{k(k-N)}{2} - \frac{N-1}{2} + \frac{(k+1)(k-N+1)}{2}} \delta_{k,\ell-1} \\
&= q^{k-(N-1)} \delta_{k,\ell-1} = q (RQ)_{k\ell} \sim (RQ)_{k\ell}
\end{aligned} \tag{A.6}$$

For  $m > 1$  one just has to replace  $X \rightarrow X^m$ , which concludes our proof of Eq. (3.30).

To prove the remaining relations in Eq. (3.34) we will need the identity<sup>13</sup>

$$Z \equiv \sum_{k=0}^{N-1} q^{\frac{(k-N/2)^2}{2}} = \sqrt{iN} \quad (\text{A.7})$$

Let us start with  $m = 1$ .

$$\begin{aligned} (XU)_{k\ell}^3 &= N^{-3/2} \sum_{i,j} q^{-\frac{k(k-N)}{2} - (k-1)i - \frac{i(i-N)}{2} - (i-1)j - \frac{j(j-N)}{2} - (j-1)\ell} \\ &= N^{-3/2} \sum_{i,j} q^{-\frac{(i+j+k-1-N/2)^2}{2} + \frac{(j+k-1-N/2)^2}{2} - \frac{k(k-N)}{2} + j - \frac{j(j-N)}{2} - (j-1)\ell} \\ &= N^{-1} (i)^{-\frac{1}{2}} \sum_j q^{(j-1)(l-k) + \frac{(1+N/2)^2}{2}} = (i)^{-\frac{1}{2}} q^{\frac{(1+N/2)^2}{2}} \delta_{k,\ell}. \end{aligned} \quad (\text{A.8})$$

The fact that we have collected a nontrivial phase (i.e. not an integer power of  $q$ ) is related to the fact that our matrices  $X$  and  $U$  are  $U(N)$  as opposed to  $SU(N)$  matrices. This could easily be remedied by a rescaling, without affecting the other relations Eq. (3.27), (3.29), and (3.30). Since  $SU(N)$  is a group, it follows that the r.h.s. of Eq. (A.8) has to be an  $SU(N)$  element also. After a suitable rescaling we thus arrive at the first relation in Eq. (3.34). For the last relation in Eq. (3.34) we calculate

$$\begin{aligned} (X^\dagger U)_{k\ell}^3 &= N^{-3/2} \sum_{i,j} q^{\frac{k(k-N)}{2} - (k-1)i + \frac{i(i-N)}{2} - (i-1)j + \frac{j(j-N)}{2} - (j-1)\ell} \\ &= N^{-3/2} \sum_{i,j} q^{\frac{(i-j-k+1-N/2)^2}{2} - \frac{(j+k-1+N/2)^2}{2} + \frac{k(k-N)}{2} + j + \frac{j(j-N)}{2} - (j-1)\ell} \\ &= N^{-1} (i)^{\frac{1}{2}} \sum_j q^{-j(-2+k+l+N) + k+l-kN - \frac{(1-N/2)^2}{2}} \\ &= (i)^{\frac{1}{2}} q^{2-Nk - \frac{(1-N/2)^2}{2}} \delta_{k,2-\ell}. \end{aligned} \quad (\text{A.9})$$

For  $N = 2$ , this is already proportional to the identity. For  $N > 2$  we square this to find

$$(X^\dagger U)_{k\ell}^6 = i q^{4-(1+N/2)^2} \delta_{k,\ell}. \quad (\text{A.10})$$

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<sup>13</sup>The fact that  $\bar{Z}Z = N$  can be inferred by considering the Discrete Fourier Transformation (DFT) of  $x_k = q^{(k-N/2)^2/2}$ . By performing the DFT and its inverse, one finds  $x_k = \bar{Z}Z/N x_k$ . We shall not prove the value of the phase in Eq. (A.7) since it will turn out to be irrelevant (see comment after Eq. (A.8)).

Finally, for  $m > 1$  we can just replace  $q \rightarrow q^m$  and observe that Eq. (A.7) still holds since  $m$  and  $N$  are coprime.

## B The moduli space for $m=0$

In this appendix we would like to calculate the moduli space on the orbifold, in the case  $m = 0$ . To this end, we calculate the scalar zero modes from the projection Eq. (3.16) and subsequently find those modes that correspond to flat directions in the potential. The potential is coming from the term

$$\mathcal{V} \sim \text{Tr}(F_{ij}F^{ij}) = 2g^{-1} \text{Tr} F_{45}^2 = -2g^{-1} \text{Tr}[A_4, A_5]^2 = 4g^{-1} \text{Tr}[A_+, A_-]^2, \quad (\text{B.1})$$

where  $g = \det g_{ij}$  and we have defined the complex scalars  $A_{\pm} = A_4 \pm iA_5$ . Notice that the hermiticity of the  $A_i$  implies the reality constraint  $A_+^\dagger = A_-$ . The orbifold boundary conditions now read:

$$A_{\pm}(pz) = \exp\left(2\pi i \left[V \mp \frac{1}{n}\right]\right) A_{\pm}(z). \quad (\text{B.2})$$

The zero modes correspond to those states where the term in the square brackets in Eq. (B.2) is integer.

To find these zero modes, note that there are  $n$  special roots that have  $V \cdot \alpha = 1/n \pmod{\mathbb{Z}}$ : the  $n-1$  simple roots that have  $k_i = 1$  in Eq. (3.16), as well as the most negative root (defined as minus the sum of all simple roots). They all belong to different irreducible representations of the subgroup  $\mathcal{H}_0$  defined in Eq. (3.17). By inspection of the remaining roots and their associated raising and lowering operators,<sup>14</sup> one can parametrize the zero modes of  $A_+$  as

$$A_+ = \begin{pmatrix} 0 & A_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_{n-1} \\ A_n & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (\text{B.3})$$

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<sup>14</sup> The positive roots of  $SU(N)$  are given by  $\{\alpha_{\ell k} = \alpha_{(\ell)} + \alpha_{(\ell+1)} + \cdots + \alpha_{(k)}, 0 \leq \ell \leq k \leq r\}$  in terms of the simple roots  $\alpha_{(i)}$ . The associated creation operator is given by  $(E_{\ell k}^+)_{ij} = \delta_{i,\ell} \delta_{j,k+1}$ .

Here the entry in the  $i$ th row and  $j$ th column is a matrix of dimension  $N_i \times N_j$ . In particular, the  $A_i$  are  $N_i \times N_{i+1}$  matrices forming the representation

$$A_i = (N_i, \bar{N}_{i+1}), \quad (\text{B.4})$$

where we have adopted a cyclic convention for the indices. One immediately calculates  $F_{+-} = [A_+, A_-]$

$$F_{+-} = \begin{pmatrix} A_1 \bar{A}_1 - \bar{A}_n A_n & & & \\ & A_2 \bar{A}_2 - \bar{A}_1 A_1 & & \\ & & \ddots & \\ & & & A_n \bar{A}_n - \bar{A}_{n-1} A_{n-1} \end{pmatrix}. \quad (\text{B.5})$$

The diagonal blocks are now square matrices of dimension  $N_i$ . The vanishing of  $F_{+-}$  is a necessary and sufficient condition for the potential

$$\mathcal{V} \sim \sum_{i=1}^n \text{Tr}(A_i \bar{A}_i - \bar{A}_{i-1} A_{i-1})^2 \quad (\text{B.6})$$

to possess a flat direction. One can always use the  $\mathcal{H}_0$  gauge symmetry to diagonalize all  $A_i \bar{A}_i$ . If there is a flat direction, then in this basis the matrices  $\bar{A}_i A_i$  must be diagonal as well. Let us define  $N_{\min} = \min\{N_i\}$ . Then all matrices  $A_i \bar{A}_i$  and  $\bar{A}_i A_i$  have at least rank  $N_{\min}$ . One concludes that if a flat direction exists, without loss of generality one can assume:

$$A_i \bar{A}_i = \bar{A}_{i-1} A_{i-1} = \text{diag}(a_1, \dots, a_{N_{\min}}, 0^{N_i - N_{\min}}), \quad (\text{B.7})$$

where the  $a_k$  are real constants. All that remains to show is that there exists a configuration  $A_i$  that fulfills Eq. (B.7). This can easily be achieved by choosing the first  $N_i$  diagonal entries of  $A_i$  equal to  $\sqrt{a_i}$  with all other entries equal to zero. A generic VEV along this flat direction breaks each  $SU(N_k)$  factor to  $SU(N_k - N_{\min})$ . To obtain the  $U(1)$  factors, it is sufficient to find the rank of the surviving subgroup, i.e., we are looking for the number of Cartan generators that satisfy

$$[A_+, H] = 0. \quad (\text{B.8})$$

To this end, note that we can view the quantity  $A_+$  as a linear map from the Cartan subalgebra to the subspace of  $\mathfrak{su}(N)$  generated by those  $E_\alpha$  that are

nonzero in  $A_+$ . Writing down the matrix corresponding to that map, it can be read off that it has rank  $N_{min}(n-1)$ . The rank-nullity theorem then states that the dimension of the kernel of that map is equal to  $N-1-N_{min}(n-1)$ , which must equal the rank of the surviving subgroup. Thus, the breaking pattern turns out to be

$$SU(N) \rightarrow \prod_{k=1}^n SU(N_k) \times U(1)^{n-1} \rightarrow \prod_{k=1}^n U(N_k - N_{min}) \times U(1)^{N_{min}-1}. \quad (\text{B.9})$$

It may be verified that the rank of this group is indeed  $N-1-N_{min}(n-1)$ . Let us summarize the conditions the different quantities in Eq. (B.9) are subject to:

$$N \geq n, \quad N_k \geq 1, \quad N_{min} = \min\{N_k\}, \quad \sum_{k=1}^n N_k = N. \quad (\text{B.10})$$

Let us now turn to shift vectors that are not of the form Eq. (3.16) but still fulfill condition (3.15). The breaking pattern will still be of the form Eq. (3.17), but now with fewer  $SU(N_k)$  factors. The simple roots and the most negative root still belong to bifundamentals. The important difference is that one or more of these bifundamentals cease to have zero modes (some  $k_i > 1$  and/or  $\sum k_i < n-1$ ). Removing one or more of the  $A_i$  from Eq. (B.5) or (B.6) clearly destroys the possibility of having flat directions. We conclude that flat directions exist if and only if  $V$  is equivalent to the form Eq. (3.16).

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